On The Association Schemes of Type II Matrices Constructed on Conference Graphs (Algebraic Combinatorics)

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On The Association Schemes of Type II Matrices Constructed on Conference Graphs

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1 Introduction

Throughout this paper, $M[i,j]$ denotes the (i,j)-entry of a matrix M and $\boldsymbol{u}[h]$ denotes the h-th entry of a vector $\boldsymbol{u}$. Let M be an $m \times n$ matrix whose entries are all nonzero. We associate an $n \times m$ matrix M^- defined by the following:

$$M^-[i,j] = \frac{1}{M[j,i]}.$$

Let I denote the identity matrix and let J denote the all ones matrix. Let $\mathrm{Mat}_n(\boldsymbol{C})$ denote the set of $n \times n$ complex matrices. $W \in \mathrm{Mat}_n(\boldsymbol{C})$ is said to be a *type II matrix* if $WW^- = nI$. It is clear that if W is a type II matrix, then the transpose ${}^tW$ of the matrix and W^- are type II matrices as well.

The definition of type II matrices was first introduced explicitly in the study of *spin models*. See [1, 6] for details.

Example 1.1 (1) Let ζ be a primitive n-th root of 1. Then the matrix $W \in \mathrm{Mat}_n(\boldsymbol{C})$ defined by $W[i,j] = \zeta^{(i-1)(j-1)}$ is a type II matrix. W is called a *cyclic type II matrix* of size n.

(2) Let α be a root of the quadratic equation $t^2 + nt + n = 0$. Then the matrix $W \in \mathrm{Mat}_n(\boldsymbol{C})$ defined by $W[i,j] = 1 + \delta_{i,j}\alpha$ is a type II matrix. W is called a *Potts type II matrix* of size n.

Let $W \in \mathrm{Mat}_n(\boldsymbol{C})$ be a type II matrix. If $S, S' \in \mathrm{Mat}_n(\boldsymbol{C})$ are permutation matrices and $D, D' \in \mathrm{Mat}_n(\boldsymbol{C})$ are nonsingular diagonal matrices, then it is easy to see that $SDWD'S'$ is also a type II matrix. We say that two type II matrices W and W' are *type II equivalent* if $W' = SDWD'S'$ for suitable choices of permutation matrices S, S' and diagonal matrices D, D'. It is clear that this defines an equivalence relation on the set of type II matrices.

For a type II matrix $W \in \mathrm{Mat}_n(\boldsymbol{C})$ and for $1 \le i, j \le n$, we define an n-dimensional column vector $\boldsymbol{u}_{i,j}^W$ by the following:

$$\boldsymbol{u}_{i,j}^W[h] = \frac{W[h,i]}{W[h,j]}.$$

Let

$$\mathcal{N}(W) = \{M \in \mathrm{Mat}_n(\boldsymbol{C}) \mid \boldsymbol{u}_{i,j}^W \text{ is an eigenvector for } M \text{ for all } 1 \le i, j \le n\}.$$

It is known that $\mathcal{N}(W)$ is the Bose-Mesner algebra of a commutative association scheme. $\mathcal{N}(W)$ is called a *Nomura algebra*. Moreover, there exists a duality map from $\mathcal{N}(W)$ to $\mathcal{N}({}^tW)$. $\mathcal{N}({}^tW)$ is called the *dual* of $\mathcal{N}(W)$.

Suzuki and the author showed that W is decomposed into a generalized tensor product if and only if $\mathcal{N}(W)$ is imprimitive [4]. We are interested in type II matrices associated with primitive association schemes. Well known examples are the following:

Example 1.2 (1) Let W be a cyclic type II matrix of size p for a prime p. Then $\mathcal{N}(W)$ is the Bose-Mesner algebra of the group scheme of the cyclic group of order p.

(2) Let W be a Potts type II matrix of size $n \ge 5$. Then $\mathcal{N}(W)$ is trivial, i.e., the Bose-Mesner algebra of the class 1 association sheme.

In this paper, we study the Nomura algebra of the type II matrix constructed on the conference graph. The *conference graph* is a strongly regular graph with parameters $(4\mu+1, 2\mu, \mu-1, \mu)$ and the eigenvalues are given as

$$k = \frac{1}{2}(v-1), \quad r = \frac{-1 \pm \sqrt{v}}{2}, \quad s = \frac{-1 \mp \sqrt{v}}{2},$$

where $v = 4\mu + 1$.

Let Γ be a formally self-dual strongly regular graph, and let A_i be the i-th adjacency matrices of Γ for $i = 0, 1, 2$. For a matrix $W = t_0A_0 + t_1A_1 + t_2A_2$ ($t_i \in \boldsymbol{C}$), Jaeger gave a condition of t_i for W to be a type II matrix (See Equation (33) in [5]); W is a type II matrix if and only if t_0, t_1, t_2 satisfy the following:

$$t_2 = t_1^{-1},$$
$$s^2 + (r+1)^2 - s(r+1)(t_1^2 + t_1^{-2}) = 1, \tag{1}$$
$$t_0 = -st_1 + (r+1)t_1^{-1} \tag{2}$$

where r, s are the nontrivial eigenvalues of Γ. We write $t_1 = t, t_2 = t^{-1}$.

Our main result is the following:

Theorem 1.1 *Let W be the type II matrix constructed on the conference graph with parameters $(4\mu+1, 2\mu, \mu-1, \mu)$. If $\mu > 2$, then $\mathcal{N}(W)$ is trivial, i.e., the Bose-Mesner algebra of the class 1 association scheme.*

2 The Entries of Type II Matrices

In this section, we consider complex numbers t_i's, which appear in the type II matrix W constructed on the conference graph.

Let $(r, s) = (\frac{-1\pm\sqrt{v}}{2}, \frac{-1\mp\sqrt{v}}{2})$ where $v = 4\mu + 1$. Note that $r + s = -1$. Then Equation (1) is equivalent to

$$t + t^{-1} = \pm s^{-1}. \tag{3}$$

Then we may regard $t \in \boldsymbol{C}$ as a root of the quadratic equation $x^2 \mp s^{-1}x + 1 = 0$. Let $\bar{t}$ be the complex conjugate of t. We have $t\bar{t} = 1$, in other words, $\bar{t} = t^{-1}$.

Consider the Garois group $G = Gal(K/\mathbf{Q})$ where $K = \mathbf{Q}(t)$. There exists $\sigma \in G$ such that $\sigma(t) = t^{-1} = \bar{t}$.

By Equation (2), we have

$$t_0 = \pm 1.$$

Here the choice of sign depends on sign of r, s.

Equation (3) has in general four solutions in t, which can be obtained from one of them by inversion or change of sign. We can obtain at most 4 kinds of type II matrices depending on the value of t for fixed r and s. We can, however, verify that if one of them is obtained from the other by inversion or change of sign of t, they are type II equivalent to each other, which means we have only one type II matrix up to type II equivalence for given r and s.

3 The Graph Description of Nomura Algebras

We restate the results of [6] about the description of Nomura algebras for type II matrices.

Let W be a type II matrix in $\mathrm{Mat}_X(\boldsymbol{C})$. Let $\Gamma(W)$ be a graph whose vertex set is $X \times X$. For two vertices (a, b) and $(c, d) \in X \times X$, we say that (a, b) is *adjacent* to (c, d) if and only if the Hermitian inner product $\langle \boldsymbol{u}_{a,b}, \boldsymbol{u}_{c,d} \rangle := \sum_{x\in X} \boldsymbol{u}_{a,b}(x)\overline{\boldsymbol{u}_{c,d}}$ is nonzero. The graph $\Gamma(W)$ is said to be a *Jones graph*. Since $\langle \boldsymbol{u}_{a,b}, \boldsymbol{u}_{c,d} \rangle$ is nonzero if and only if $\langle \boldsymbol{u}_{c,d}, \boldsymbol{u}_{a,b} \rangle$ is nonzero we obtain an undirected graph $\Gamma(W)$.

Let $C_0, C_1, \ldots, C_d$ denote the connected components of a Jones graph Γ. Let A_i be a matrix in $\mathrm{Mat}_X(\boldsymbol{C})$ with (a,b)-entry equal to 1 if $(a,b) \in C_i$ and to 0 otherwise. Let $V = \boldsymbol{C}^X$, and let $V_i := \mathrm{Span}\{\boldsymbol{u}_{a,b} \mid (a,b) \in C_i\}$. It is easy to see that V is decomposed into an orthogonal direct sum of $V_0, \ldots, V_d$. Let E_i be the projection of V to V_i for $i = 0, \ldots, d$.

Proposition 3.1 ([6] Theorem 5) (1) *The set* $\{A_i \mid i = 0, 1, \ldots, d\}$ *is the basis of Hadamard idempotents of* $\mathcal{N}({}^t W)$.

(2) *The set* $\{E_i \mid i = 0, 1, \ldots, d\}$ *is the basis of primitive idempotents of* $\mathcal{N}(W)$.

In order to prove that $\mathcal{N}(W)$ is trivial, it suffices to show that the number of the connected components of $\Gamma(W)$ is equal to 2.

It is trivial that $\{(a,a) \in X \times X \mid a \in X\}$ becomes a connected component of $\Gamma(W)$. We write $C_0 := \{(a,a) \in X \times X \mid a \in X\}$.

Proposition 3.2 *Let W be a type II matrix of size $|X| \geq 5$. If $\langle \boldsymbol{u}_{a,b}, \boldsymbol{u}_{c,d} \rangle$ is nonzero where $a, b, c, d \in X$ are all distinct, then $\mathcal{N}(W)$ is trivial.*

4 Proof of Theorem 1.1

Let W be the type II matrix constructed on the conference graph with parameters $(4\mu+1, 2\mu, \mu-1, \mu)$. Let X be the vertex set of the graph with order $v = 4\mu + 1$. In this section, we show that $\langle \boldsymbol{u}_{a,b}, \boldsymbol{u}_{c,d} \rangle$ is nonzero for distinct $a, b, c, d \in X$ where $v > 9$, which implies that Theorem 1.1 holds.

Let t satisfy Equation (3). It is easy to see that $\langle \boldsymbol{u}_{a,b}, \boldsymbol{u}_{c,d} \rangle$ is a linear combination of $1, t, t^{-1}, t^2, t^{-2}, t^3, t^{-3}, t^4, t^{-4}$ over $\boldsymbol{Q}$. We can see that t, t^{-1}, t^3, t^{-3} appear if and only if $x = a, b, c,$ or d. Set $U_W(t, t^{-1}) := \sum_{x=a,b,c,d} \boldsymbol{u}_{a,b}[x]\overline{\boldsymbol{u}_{c,d}[x]}$, which is a polynomial in t, t^{-1}. Hence we have the following:

$$\langle \boldsymbol{u}_{a,b}, \boldsymbol{u}_{c,d} \rangle = U_W(t, t^{-1}) + l_1 t^2 + l_2 t^{-2} + m_1 t^4 + m_2 t^{-4} + n,$$

where $4 + l_1 + l_2 + m_1 + m_2 + n = v$. Then $\pm U_W(t, t^{-1})$ is a linear combination of t, t^{-1}, t^3, t^{-3} in which the coefficients sum to 4. The sign depends on that of t_0.

Let $r = \frac{-1 \pm \sqrt{v}}{2}$. Since $t + t^{-1} = \pm(r+1)^{-1}$ and $t_0 = (r+1)(t + t^{-1})$, we can choose plus sign for $t + t^{-1}$ so that $t_0 = 1$ without loss of generality. We will show that $\langle \boldsymbol{u}_{a,b}, \boldsymbol{u}_{c,d} \rangle$ is nonzero by way of contradiction. Assume $\langle \boldsymbol{u}_{a,b}, \boldsymbol{u}_{c,d} \rangle = 0$. Since $\langle \boldsymbol{u}_{a,b}, \boldsymbol{u}_{c,d} \rangle$ can be regarded as a polynomial in t, t^{-1} over $\boldsymbol{Q}$, we may write $f(t, t^{-1}) = \langle \boldsymbol{u}_{a,b}, \boldsymbol{u}_{c,d} \rangle$. As we have seen before, there exists

$\sigma \in G = Gal(K/\mathbf{Q})$ such that $\sigma(t) = \bar{t} = t^{-1}$. Hence $f(t^{-1}, t) = \sigma(f(t, t^{-1})) = 0$. Therefore we have $f(t, t^{-1}) + f(t^{-1}, t) = 0$, which is equivalent to

$$(l_1 + l_2)(t^2 + t^{-2}) + (m_1 + m_2)(t^4 + t^{-4}) + 2n + U_W(t, t^{-1}) + U_W(t^{-1}, t) = 0.$$

Set $l = l_1 + l_2$ and $m = m_1 + m_2$. Then we have

$$l(t^2 + t^{-2}) + m(t^4 + t^{-4}) + 2n + U_W(t, t^{-1}) + U_W(t^{-1}, t) = 0, \cdots (*)$$

where $4 + l + m + n = v$. $U_W(t, t^{-1}) + U_W(t^{-1}, t)$ is one of the following:

$4(t+t^{-1}), 4(t^3+t^{-3}), 2(t+t^{-1})+2(t^3+t^{-3}), (t+t^{-1})+3(t^3+t^{-3}), 3(t+t^{-1})+(t^3+t^{-3})$.

Note that

$$\begin{aligned} t^2 + t^{-2} &= (t + t^{-1})^2 - 2, \\ t^3 + t^{-3} &= (t + t^{-1})^3 - 3(t + t^{-1}), \\ t^4 + t^{-4} &= (t + t^{-1})^4 - 4(t + t^{-1})^2 + 2. \end{aligned}$$

Equation $(*)$ can be written as follows:

$$m(t+t^{-1})^4+(l-4m)(t+t^{-1})^2+2m+2n-2l+U_W(t,t^{-1})+U_W(t^{-1},t) = 0. \cdots (**)$$

Let $X = t + t^{-1}$. Then the left hand side of Equation $(**)$ can be expressed as a polynomial in X with degree at most 4, which is denoted by $g(X)$, i.e.,

$$g(X) = mX^4 + \alpha X^3 + (l - 4m)X^2 + \beta X + 2m + 2n - 2l,$$

where $\alpha X^3 + \beta X = U_W(t, t^{-1}) + U_W(t^{-1}, t)$.

Note that

$$\begin{aligned} 4(t + t^{-1}) &= 4X, \\ 4(t^3 + t^{-3}) &= 4(X^3 - 3X) = 4X^3 - 12X, \\ 2(t + t^{-1}) + 2(t^3 + t^{-3}) &= 2X + 2(X^3 - 3X) = 2X^3 - 4X, \\ (t + t^{-1}) + 3(t^3 + t^{-3}) &= X + 3(X^3 - 3X) = 3X^3 - 8X, \\ 3(t + t^{-1}) + (t^3 + t^{-3}) &= 3X + (X^3 - 3X) = X^3. \end{aligned}$$

Hence the value of (α, β) is given as follows:

$U_W(t, t^{-1}) + U_W(t^{-1}, t)$	(α, β)
$4(t + t^{-1})$	$(0, 4)$
$4(t^3 + t^{-3})$	$(4, -12)$
$2(t + t^{-1}) + 2(t^3 + t^{-3})$	$(2, -4)$
$(t + t^{-1}) + 3(t^3 + t^{-3})$	$(3, -8)$
$3(t + t^{-1}) + (t^3 + t^{-3})$	$(1, 0)$

Lemma 4.1 *Let v be a square. Let W be the type II matrix constructed on the conference graph of order $v = v'^2 > 9$ where v' is an integer. Then $\langle \boldsymbol{u}_{a,b}, \boldsymbol{u}_{c,d} \rangle$ is nonzero for any distinct $a, b, c, d \in X$.*

Proof. The minimal polynomial of $t + t^{-1}$ is $h(X) = X - \frac{2}{1 \pm v'}$ for $t + t^{-1} = \frac{2}{1 \pm \sqrt{v}}$. The constant part of the remainder of $g(X)/h(X)$ is

$$2m + 2n - 2l + \frac{2}{1 \pm v'}(\beta + \frac{2}{1 \pm v'}(l - 4m + \frac{2}{1 \pm v'}(\alpha + \frac{2m}{1 \pm v'}))),$$

which is equivalent to

$$2(m + n - l) + \frac{2\beta}{1 \pm v'} + \frac{4(l - 4m)}{(1 \pm v')^2} + \frac{8\alpha}{(1 \pm v')^3} + \frac{16m}{(1 \pm v')^4}.$$

The constant part of the remainder must be zero if $\langle \boldsymbol{u}_{a,b}, \boldsymbol{u}_{c,d} \rangle = 0$. Hence we have

$$m + n - l + \frac{\beta}{1 \pm v'} + \frac{2(l - 4m)}{(1 \pm v')^2} + \frac{4\alpha}{(1 \pm v')^3} + \frac{8m}{(1 \pm v')^4} = 0.$$

Since $4 + l + m + n = v'^2$, we have $m + n - l = v'^2 - 4 - l$. So the above equation is equivalent to

$$v'^2 - 4 - 2l + \frac{\beta}{1 \pm v'} + \frac{2(l - 4m)}{(1 \pm v')^2} + \frac{4\alpha}{(1 \pm v')^3} + \frac{8m}{(1 \pm v')^4} = 0.$$

Multiplying $(1 \pm v')^4$, we have

$$(v'^2 - 4 - 2l)(1 \pm v')^4 + \beta(1 \pm v')^3 + 2(l - 4m)(1 \pm v')^2 + 4\alpha(1 \pm v') + 8m = 0.$$

This is equivalent to

$$(v'^2 - 4)(1 \pm v')^4 + \beta(1 \pm v')^3 + 4\alpha(1 \pm v') - 2l((1 \pm v')^2 - 1) - 8m((1 \pm v')^2 - 1) = 0.$$

Therefore we have

$$(v' + 2)(v' - 2)(1 \pm v')^4 + \beta(1 \pm v')^3 + 4\alpha(1 \pm v') - 2lv'(v' \pm 2) - 8mv'(v' \pm 2) = 0.$$

Set $B = \beta(1 \pm v')^3 + 4\alpha(1 \pm v')$. Then the above equation is equivalent to

$$(v' + 2)(v' - 2)(1 \pm v')^4 + B - 2lv'(v' \pm 2) - 8mv'(v' \pm 2) = 0.$$

So B must be divisible by $(v' \pm 2)$. However we have the following:

(α,β)	B
$(0,4)$	$\pm 4(v'\pm 2)(v'^2\pm v'+1)-4$
$(4,-12)$	$\mp 4(v'\pm 2)(3v'^2\pm 3v'-1)-4$
$(2,-4)$	$\mp 4(v'\pm 2)(4v'^2\pm 4v'-1)-4$
$(3,-8)$	$\mp 4(v'\pm 2)(2v'^2\pm 2v'-1)-4$
$(1,0)$	$\mp 4(v'\pm 2)-4$

If B is divisible by $v'\pm 2$, then 4 will be divisible by $v'\pm 2$. So

$$v'\pm 2=\pm 1,\pm 2,\pm 4.$$

Hence

$$v'=\pm 1,\pm 3,0,\pm 4,\pm 2,\pm 6.$$

Since $v=v'^2\equiv 1 \pmod 4$ and $v>1$, $v'\neq 0,\pm 1,\pm 2,\pm 6$. It is only possible $v'=\pm 3$. Therefore B is not divisible by $v'\pm 2$ except for the case $v=v'^2=9$, which is a contradiction. Hence $\langle \boldsymbol{u}_{a,b},\boldsymbol{u}_{c,d}\rangle$ is nonzero whenever $v>9$. ∎

Lemma 4.2 *Let v be a nonsquare. Let W be the type II matrix constructed on a conference graph of order $v>5$. Then $\langle \boldsymbol{u}_{a,b},\boldsymbol{u}_{c,d}\rangle$ is nonzero for any distinct $a,b,c,d\in X$.*

Proof. The minimal polynomial of $t+t^{-1}$ is $h'(X)=X^2-\frac{4}{1-v}X+\frac{4}{1-v}$ for $t+t^{-1}=\frac{2}{1\pm\sqrt{v}}$. The constant part of the remainder of $g(X)/h'(X)$ is

$$2m+2n-2l-\frac{4}{1-v}(l-4m-\frac{4m}{1-v}+\frac{4}{1-v}(\alpha+\frac{4m}{1-v})),$$

which is equivalent to

$$2(m+n-l)-\frac{4}{1-v}\{l-4m-\frac{4m}{1-v}+\frac{16m}{(1-v)^2}+\frac{4\alpha}{1-v}\}.$$

Set $B'=4\alpha$. Then we have the following:

$U_W(t,t^{-1})+U_W(t^{-1},t)$	B'
$4(t+t^{-1})$	0
$4(t^3+t^{-3})$	16
$2(t+t^{-1})+2(t^3+t^{-3})$	8
$(t+t^{-1})+3(t^3+t^{-3})$	12
$3(t+t^{-1})+(t^3+t^{-3})$	4

The constant part of the remainder must be zero if $\langle \boldsymbol{u}_{a,b},\boldsymbol{u}_{c,d}\rangle=0$. Hence we have

$$2(m+n-l)-\frac{4}{1-v}\{l-4m-\frac{4m}{1-v}+\frac{16m}{(1-v)^2}+\frac{B'}{1-v}\}=0,$$

Multiplying $\frac{1}{2}(1-v)^3$, we get

$$(m+n-l)(1-v)^3 - 2(l-4m)(1-v)^2 + 8m(1-v) - 32m - 2B'(1-v) = 0.$$

Since $4+l+m+n=v$, we can eliminate n by putting $m+n-l = v-4-2l$. Hence we have

$$(v-4-2l)(1-v)^3 - 2(l-4m)(1-v)^2 + 8m(1-v) - 32m - 2B'(1-v) = 0.$$

We can rewrite the above equation with respect to l, m as follows:

$$(v-4)(v-1)^3 - 2l(v-2)(v-1)^2 - 8m(v^2-3v-2) - 2B'(v-1) = 0. \cdots (***)$$

Since $v = 4\mu + 1$, where μ is a positive integer, we have

$$v - 1 = 4\mu,$$

$$(v-1)^2 = 4\mu(v-1),$$

$$(v-1)^3 = 16\mu^2(v-1).$$

Note that B' is even. Therefore $(v-1)^3$, $(v-1)^2$, $2B'(v-1)$ are divisible by $4(v-1)$, although v^2-3v-2 is not. So $4(v-1)$ must divide $8m$, in other words, 4μ must divide $2m$. Hence there exists a non-negative integer a such that $m = 2\mu a$. Since $4+l+m+n = v = 4\mu+1$, we have $m < 4\mu - 3 < 4\mu$. So $2\mu a < 4\mu$, or equivalently $a < 2$. Hence $a = 1$, i.e., $8m = 16\mu = 4(v-1)$. By Equation $(***)$, we have the following:

$$\begin{aligned} l &= \frac{1}{2(v-2)(v-1)^2}\{(v-4)(v-1)^3 - 4(v-1)(v^2-3v-2) - 2B'(v-1)\} \\ &= \frac{1}{2(v-2)(v-1)}\{(v-4)(v-1)^2 - 4(v^2-3v-2) - 2B'\} \\ &= \frac{1}{2(v-2)(v-1)}(v^3 - 10v^2 + 21v + 4 - 2B') \\ &= (v-7)(v^2-3v+2) - \frac{v-9+B'}{(v-2)(v-1)}. \end{aligned}$$

Since $v = 4\mu+1$ is a positive nonsquare, we have

$$v = 5, 13, 17, \ldots.$$

Note that B' is a non-negative integer. Then we have

$$v - 9 + B' > 0 \text{ and } (v-2)(v-1) > 0 \text{ if } v > 5.$$

Moreover if $v > 5$, we have

$$\begin{aligned}(v-2)(v-1)-(v-9+B') &= v^2-4v+B' \\ &= v(v-4)+11+B' \\ &> 0.\end{aligned}$$

So $\frac{v-9+B'}{(v-2)(v-1)}$ is not an integer if $v > 5$, which contradicts the fact that l is an integer.

Therefore we have a contradiction if $v > 5$. This completes the proof. ■.

Proof of Theorem 1.1 By Proposition 3.2,Lemma 4.1, and Lemma 4.2, it is clear. ■

Remarks.

(1) The type II matrix constructed on the conference graph of order 5 is type II equivalent to the cyclic type II matrix of size 5, and the Nomura algebra is the Bose-Mesner algebra of the group scheme of the cyclic group $\boldsymbol{C}_5$.

(2) If r is negative, the type II matrix W constructed on the conference graph of order 9 is type II equivalent to the tensor product of 2 copies of Potts type II matrices of size 3, and $\mathcal{N}(W)$ is the Bose-Mesner algebra of the group scheme of $\boldsymbol{C}_3 \otimes \boldsymbol{C}_3$. If r is positive, $\mathcal{N}(W)$ is trivial.

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